Analysis SEP 2026: Advanced Calculus (Part II)

1 Warm-up

Suppose $s_k \to s$. Prove that $\frac{1}{n}\sum_{k=1}^{n} s_k \to s$.

2 Exercises

Problem 8.21.1 Let $f \in C^1([0,1])$. Show that for every $\varepsilon > 0$, there exists a polynomial p such that

$$\|f - p\|_\infty + \|f' - p'\|_\infty < \varepsilon.$$

Problem 1.24.1 For a sequence (a_k) let $s_n = \sum_{k=1}^{n} a_k$ and $\sigma_L = \frac{1}{L}\sum_{n=1}^{L} s_n$. We say that the series $\sum_{k=1}^{\infty} a_k$ is *Cesàro summable* to S if $\lim_{L\to\infty} \sigma_L = S$.

(i) Prove: $s_n - \sigma_n = \frac{(n-1)a_n + (n-2)a_{n-1} + \cdots + a_2}{n}$.

(ii) Prove: If $\sum_{k=1}^{\infty} a_k$ is Cesáro summable to S and if $\lim_{k\to\infty} k a_k = 0$ then the series $\sum_{k=1}^{\infty} a_k$ converges and we have $\sum_{k=1}^{\infty} a_k = S$.

Problem 1.25.1 Let $\{a_n\}$ be a sequence of complex numbers and suppose that $\sum_{n=1}^{\infty} a_n$ converges. Prove that

$$\lim_{n\to\infty} \sum_{k=1}^{n} (1 - \frac{k}{n}) a_k = \sum_{n=1}^{\infty} a_n.$$

Problem 8.20.2 Show that $\lim_{x\to 0} \int_0^\infty \sin\big(t^2 + xt\big)\, dt$ exists, where the integral is evaluated as an improper integral.

Problem 1.25.3 Let $\mathbb{R}^{n\times n}$ be the vector space of all real $n \times n$ matrices. Let $\|\cdot\|$ be a norm on $\mathbb{R}^{n\times n}$ and consider $\mathbb{R}^{n\times n}$ as a normed vector space. For $A \in \mathbb{R}^{n\times n}$ denote by $\operatorname{tr}(A)$ the trace of A (i.e. the sum of the diagonal elements). Prove that there are neighborhoods U and V of the identity matrix I such that for every $B \in V$ there exists a unique $A \in U$ such that

$$\frac{1}{n} \operatorname{tr}(A) A^5 = B.$$

Hint from ya gurl: The statement is somewhat reminiscent of the Banach fixed point theorem, but note there is not a fixed point in the given identity. However, the Banach fixed point theorem *does* imply a theorem that you could use to prove the statement.

Problem 1.20.3 Show that $\int_0^\infty \frac{\sin(x)}{x^{2/3}}\, dx$ converges. Determine if

$$\int_1^\infty \frac{\sin(x)}{x^{2/3} + \sin(x)}\, dx$$

converges. Hint: Use Taylor expansion.

Problem 8.24.2 Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that the limit

$$\lim_{n\to\infty} \int_0^1 f(x)\,|\sin(2\pi n x)|\,dx$$

exists and determine its value.

Hint from ya gurl: The fact that $[0,1] = \bigcup_{k=0}^{n-1} [k/n, (k+1)/n]$ will help substantially.

3 Homework :-)

1. Prove that the implicit function theorem and the inverse function theorem are equivalent statements (i.e., one implies the other, and vice versa).
2. Write out as many of the various function space density theorems as you remember off the top of your head (e.g., simple functions are dense in $L^p(X)$ for $1 \le p \le \infty$, Stone-Weierstrass, etc...). Make a comprehensive list (or flashcards) of the theorems. *Note:* A lot of the important ones you should know are scattered throughout the recommended texts (Folland, Rudin, etc.); an alternate route would be to ask your favorite LLM what theorems fall into this category—or reference this conversation that I already had with ChatGPT—and then find textbook citations for all these claims.
3. If you haven't already, start making a study sheet or flashcards of all the various theorems or tricks we have encountered so far in the SEP.